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VOLUME XXIX NUMBER 1

THE

JOURNAL OF GEOLOGY

JANUARY-FEBRUARY 1921

THE MECHANICAL INTERPRETATION OF JOINTS

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PART II

OUTLINE

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I. MOHR'S THEORY OF RUPTURE[1]

BY LOUIS BRAND

Let P denote a point of a body at which the state of stress is to be investigated. For this purpose imagine a sphere of infinitesimal radius described about P as center (Fig. 10); then the totality of the stresses at all the surface elements of the sphere constitute the state of stress at the point P. A surface element of the sphere may be specified by means of the vector radius to the element. The surface r thus means an infinitely small plane surface tangent to the sphere at the end of r. In the following, the stresses shown acting upon the sphere are those due to outside matter. Since we assume that the stress distribution within the body is continuous, the stresses that correspond to the two surface elements at the end of a diameter of the sphere are therefore numerically equal and opposite in sign, except for negligibly small quantities.

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[1] This account of Mohr's theory of rupture was written at the writer's request by Professor Louis Brand, of the University of Cincinnati, to whom he wishes herewith to express his sincere thanks.

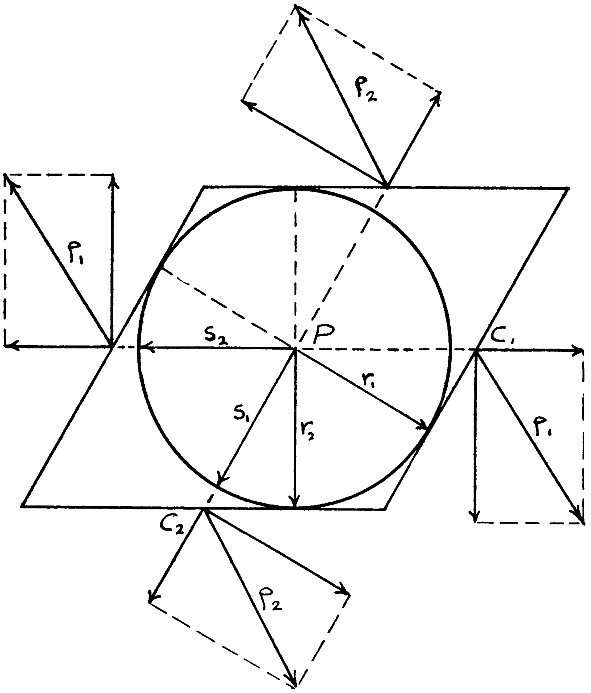

FIG. 10

We shall now obtain the relation between the unit stresses ρ_1, ρ_2 at the two surfaces r_1, r_2. Form a parallelepiped with three pairs of planes tangent to the sphere, one pair normal to r_1, another to r_2, while a third pair is normal to a radius r_3 which is perpendicular to both r_1 and r_2. Figure 10 shows the projection of this parallelepiped upon a plane through P normal to r_3. Both faces normal to r_3 project into a rhombus. The other four faces are rectangles of equal area, dA, and project into lines. Now resolve the stresses ρ_1 into three components: parallel to r_2, parallel to r_3, and normal to r_2, r_3. Similarly resolve the stresses ρ_2 into three components: parallel to r_1, parallel to r_3, and normal to r_1, r_3. Of these components, only the first named of each set, namely,

$$\rho_1 \cos(\rho_1 r_2),\ \rho_2 \cos(\rho_2 r_1),$$

have moments about r_3. Hence for equilibrium as regards rotation about r_3, we have

$$2dA \cdot \rho_1 \cos(\rho_1 r_2) \cdot \overline{PC_1} = 2dA \cdot \rho_2 \cos(\rho_2 r_1) \cdot \overline{PC_2};$$

or, since $\overline{PC_1} = \overline{PC_2}$,

$$\rho_1 \cos(\rho_1 r_2) = \rho_2 \cos(\rho_2 r_1). \qquad (1)$$

The angles indicated are those between the stresses and *outwardly directed radii.* The fundamental equation (1) may be stated as follows:

If ρ_1, ρ_2 are unit stresses at the surface elements r_1, r_2, the projection of ρ_1 upon r_2 is equal to the projection of ρ_2 upon r_1.

We proceed to put equation (1) in a more usable form. Let us regard angles in the plane of r_1, r_2 as positive when clockwise; and let s_1, s_2 denote radii in this plane 90° *ahead* of r_1, r_2 respectively. Now resolve ρ_1 into components as follows: (1) $\sigma_1 = \rho_1 \cos(\rho_1 r_1)$, parallel to r_1; (2) $\tau_1 = \rho_1 \cos(\rho_1 s_1)$, parallel to s_1; (3) parallel to r_3. Also resolve ρ_2 into the components: (1) $\sigma_2 = \rho_2 \cos(\rho_2 r_2)$, parallel to r_2; (2) $\tau_2 = \rho_2 \cos(\rho_2 s_2)$, parallel to s_2; (3) parallel to r_3. The *normal stresses* σ_1, σ_2 (i.e., normal to their surface elements) are positive when directed outward from P, or when they are *tensions.* The components τ_1, τ_2 of the *shearing stresses* are positive when they produce clockwise moments about r_3. We now transform the cosines in (1) by means of the formula for the angle between two directions in space:

$$\begin{aligned}
\cos(\rho_1 r_2) &= \cos(\rho_1 r_1)\cos(r_2 r_1) + \cos(\rho_1 s_1)\cos(r_2 s_1) + \cos(\rho_1 r_3) \cdot \cos(r_2 r_3)\\
&= \cos(\rho_1 r_1)\cos(r_1 r_2) + \cos(\rho_1 s_1)\sin(r_1 r_2),\\
\cos(\rho_2 r_1) &= \cos(\rho_2 r_2)\cos(r_1 r_2) + \cos(\rho_2 s_2)\cos(r_1 s_2) + \cos(\rho_2 r_3)\cos(r_1 r_3)\\
&= \cos(\rho_2 r_2)\cos(r_1 r_2) - \cos(\rho_2 s_2)\sin(r_1 r_2).
\end{aligned}$$

Substituting the values in (1) and noting the above values for σ_1, σ_2, τ_1, τ_2, we have

$$\sigma_1 \cos (r_1r_2)+\tau_1 \sin (r_1r_2)=\sigma_2 \cos (r_1r_2)-\tau_2 \sin (r_1r_2),$$

or

$$(\sigma_2-\sigma_1) \cos (r_1r_2)=(\tau_1+\tau_2) \sin (r_1r_2). \tag{2}$$

As a first application of this equation we have

$$\tau_1+\tau_2=0 \quad \text{when} \quad (r_1r_2)=90^\circ. \tag{3}$$

In this case the shearing stresses are numerically equal but opposite in sign.

Next, let r_0 be any fixed radius in the plane r_1, r_2; and write

$$\phi=(r_0r_1),\ \phi+\Delta\phi=(r_0r_2);\ \sigma_2=\sigma_1+\Delta\sigma,\ \tau_2=\tau_1+\Delta\tau.$$

Then (2) assumes the form

$$\Delta\sigma \cos \Delta\phi=(2\tau_1+\Delta\tau) \sin \Delta\phi;$$

dividing through by $\Delta\phi$ and letting $\Delta\phi$ approach zero, we obtain

$$\frac{d\sigma}{d\phi}=2\tau, \tag{4}$$

the subscript being no longer needed. From this equation it appears that σ increases with ϕ when τ has the direction of increasing ϕ.

If σ is not constant over the sphere it must reach a minimum σ_x and a maximum σ_z at certain diameters which we denote by x and z respectively. At these points $\frac{d\sigma}{d\phi}=0$ for *all* diametral planes; hence from (4) the *total* shear vanishes at x and z. Also, from (2), $\cos (xz)=0$, so that x and z are perpendicular. Again, if y is a third diameter, perpendicular to both x and z, the components of shear in both yx and yz planes vanish by virtue of (3), and hence the *total* shear for y is also zero. The three mutually perpendicular diameters x, y, z, are called the *principal axes*, the planes xy, yz, zx, the *principal planes*, and the stresses σ_x, σ_y, σ_z, the *principal stresses* at the point P. When the principal stresses are unequal the notation has been chosen so that $\sigma_x < \sigma_y < \sigma_z$.

We now choose positive directions on the principal axes, and denote the positive principal radii by r_x, r_y, r_z. Then, if ρ is the stress at any surface r, we have from (1)

$$\left.\begin{aligned} \rho \cos(\rho r_x) &= \sigma_x \cos(r r_x), \\ \rho \cos(\rho r_y) &= \sigma_y \cos(r r_y), \\ \rho \cos(\rho r_z) &= \sigma_z \cos(r r_z). \end{aligned}\right\} \tag{5}$$

The components of the stress at any surface r are thus given in terms of the principal stresses and the direction cosines of r. From these components we may compute the normal stress, σ, at r:

$$\sigma = \rho \cos(\rho r) = \sigma_x \cos^2(r r_x) + \sigma_y \cos^2(r r_y) + \sigma_z \cos^2(r r_z). \tag{6}$$

The equations (5) show, moreover, that the stresses over the sphere are symmetrically distributed with respect to each of the three principal planes. It is sufficient, therefore, to know the stress distribution in one of the octants into which the principal planes divide the sphere.

We shall now examine the stress distribution in the principal planes. Let r be a radius in the xz plane making an angle $\phi = (r r_z)$ with the positive z-axis. Since $(r r_y) = 90°$, we have from (5) that $(\rho r_y) = 90°$; hence the stresses in a principal plane lie entirely in that plane. From (6) we have for the normal stress at r

$$\sigma = \sigma_x \sin^2 \phi + \sigma_z \cos^2 \phi = \frac{\sigma_x + \sigma_z}{2} + \frac{\sigma_z - \sigma_x}{2} \cos 2\phi. \tag{7}$$

To obtain τ we put $\sigma_2 = \sigma_z$, $\tau_2 = \tau_z = 0$, $r_2 = r_z$ in (2); then, by virtue of (7),

$$\tau = (\sigma_z - \sigma) \frac{\cos \phi}{\sin \phi} = (\sigma_z - \sigma_x) \sin \phi \cos \phi = \frac{\sigma_z - \sigma_x}{2} \sin 2\phi. \tag{8}$$

If σ, τ are regarded as rectangular co-ordinates, equations (7) and (8) are the parametric equations of a circle of center $[(\sigma_x + \sigma_z)/2, 0]$ and radius $(\sigma_z - \sigma_x)/2$. We thus have the following graphical representation of the stress distribution in the xz plane: Lay off

$\overline{OX} = \sigma_x$, $\overline{OZ} = \sigma_z$ on the axis of abscissas (Fig. 11), and draw a circle having $\overline{XZ}$ as diameter. This circle has the center and radius specified above; and the abscissa and ordinate of any point R of the circle represent the normal and shearing stress, σ, τ, respectively, which correspond to a radius r of the sphere in the xz plane and making an angle ϕ with r_z equal to one-half of the central angle ZCR.

If we lay off $\overline{OY} = \sigma_y$, a similar argument shows that the stress distributions on the xy and yz planes are represented by the co-ordinates of the points forming the circles on $\overline{XY}$ and $\overline{YZ}$ as diameters. It may also be shown that for any radius of the sphere not in the principal planes, the corresponding stresses σ, τ, are represented by the co-ordinates of points within the region bounded by all three of the circles XY, YZ, XZ. *The circle XZ therefore passes through all the points which have the greatest τ for any given σ occurring in the stress distribution.* This circle plays an important rôle in Mohr's theory of rupture, and is called by Mohr the *principal circle* in the graphic representation of the state of stress.

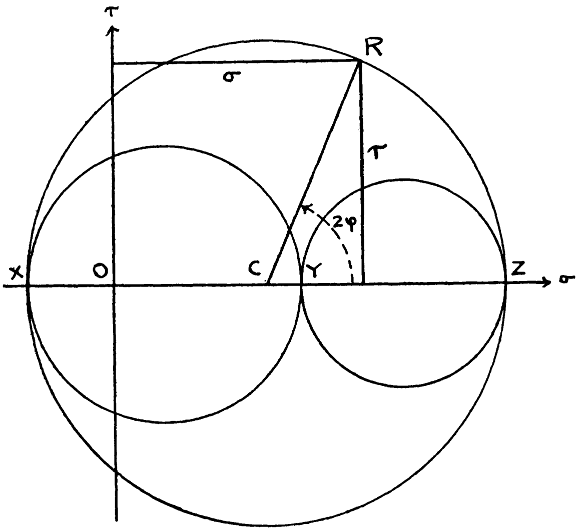

Fig. 11

Mohr's theory of rupture deals with cases in which the failure of the material is assumed to be due to the sliding of one layer over another in certain planes called *shearing planes.* The fundamental postulate of this theory may be stated as follows: *The ultimate strength of the material is determined by the greatest shearing stress in the shearing planes, this limiting shear itself depending upon the normal stress in the planes.* The limiting shear τ_m is thus assumed to be some function of the normal stress in the plane, $\tau_m = f(\sigma)$, but the nature of this dependence is not specified by the theory. The curve, however, which represents the relation $\tau_m = f(\sigma)$ has certain properties which may be inferred from Mohr's graphic representation of stress distributions. In this representation a point S_1, corresponding to a shearing plane (Fig. 12), must lie on a principal circle, since the points of this circle have the greatest τ for a given value of σ. From this fact we may draw two important conclusions:

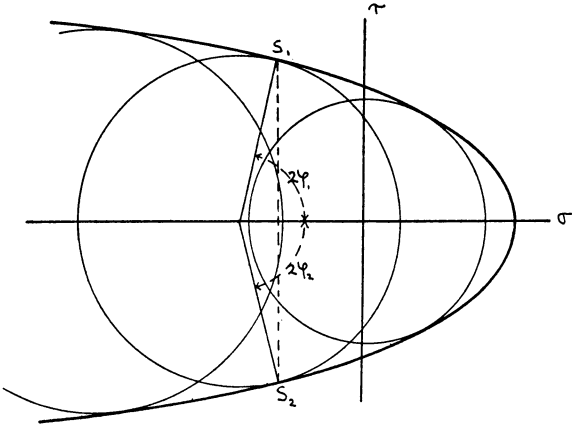

FIG. 12

1. Since a principal circle is completely determined by the two extreme principal stresses, σ_x, σ_z, the ultimate strength of a material that fails by slippage must be entirely independent of the intermediate principal stress σ_y.

2. Since points on a principal circle correspond to planes normal to the xz plane, the shearing planes at any point must pass through the y-axis.

Furthermore, a point on a principal circle corresponding to a shearing plane must lie on the envelope of all the principal circles *representing ultimate states of stress.* For this envelope is the locus of all points, which, for a given σ, have the greatest τ for the stress distributions in question. The envelope thus represents the relation $\tau_m = f(\sigma)$ between maximum shear and normal stress. As all principal circles have their centers on the σ-axis, their envelope is symmetric with respect to this axis, and touches each circle in two points, S_1, S_2, corresponding to the two shearing planes for the state of stress represented by the circle. Referring to Figure 12, we see that the shearing planes corresponding to S_1 and S_2 make equal angles with the zy plane: $\phi_1 = \phi_2$. The acute angle between the planes is therefore equal to $2\phi_1$, and is bisected by the principal plane zy.

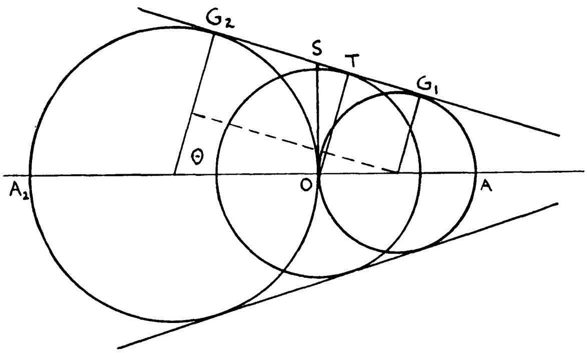

FIG. 13

Experiment has shown that the envelopes are roughly parabolic in shape. They presumably cut the $+\sigma$-axis at the point-circle representing rupture due to uniform all-sided tension ($\sigma_x = \sigma_y = \sigma_z$). Mohr also infers from the apparent impossibility of crushing a substance by applying uniform all-sided pressure, that if the envelope cuts the $-\sigma$-axis at all, it does so at an excessive distance from the origin.

If the ultimate tensile and compressive strength of a material are known, say σ_1 and σ_2, we may construct the principal circles corresponding to these limiting states, namely,

$$\sigma_x = \sigma_y = 0, \quad \sigma_z = \sigma_1; \qquad \sigma_x = -\sigma_2, \quad \sigma_y = \sigma_z = 0.$$

Thus in Fig. 13 we lay off $\overline{OA}_1=\sigma_1$, $\overline{OA}_2=-\sigma_2$, and draw the principal circles on $\overline{OA}_1$ and $\overline{OA}_2$. For states in which the principal stresses do not considerably exceed σ_1 and $-\sigma_2$, we may regard the common external tangents to these circles as representing approximately a portion of the envelope. For the region in which this approximation is allowable, the angle between the shearing planes is constant and given by

$$\cos\theta=\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}=\frac{\kappa-1}{\kappa+1}, \quad \kappa=\frac{\sigma_2}{\sigma_1}. \tag{9}$$

The angle θ is thus independent of the particular state of stress and depends only upon the ratio of ultimate compressive to ultimate tensile stress. According as $\sigma_2 > \; = \; < \sigma_1$, θ will be an acute, right, or obtuse angle. For most materials $\sigma_2>\sigma_1$; values of θ corresponding to different values of κ in this case are given in the following table:

$\kappa=\frac{\sigma_2}{\sigma_1}$	1	1.5	2	3	4	5	10	20
θ	90°	78°	71°	60°	53°	49°	35°	25°

The principal circle for the state in which $\sigma_x=-\sigma_z$, that is, a circle about O as center and tangent to the envelope tangents, determines the ultimate torsional strength: $\rho_3=\overline{OT}$. The ultimate shearing strength is given by $\tau_4=\overline{OS}$; for this ordinate represents the maximum shear for zero normal stress. From the geometry of the figure,

$$\rho_3=\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}, \quad \tau_4=\tfrac{1}{2}\sqrt{\sigma_1\sigma_2}\,.$$

When σ_1 and σ_2 are nearly equal we have $\rho_3=\tau_4=\frac{1}{2}\sigma$, a result in concordance with many old and modern experiments and not satisfactorily explained by the earlier theories of rupture.

II. MOHR'S THEORY APPLIED TO EXPERIMENTAL DATA

Mohr's formula, $\cos\theta=\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}$, connects the angle of shearing with two important physical constants, the ultimate tensile strength and the crushing strength.

Within certain limits, that is to say, in so far as the curve $\tau_m=f(\sigma)$ between the points G_1 and G_2 approaches a straight line, it confirms

Hartmann's observation that the angle of shearing is independent of the nature and intensity of the stresses involved.

It also brings out clearly, in a qualitative way, the very different attitude of the planes of shearing in brittle and ductile substances. This becomes apparent when we introduce into it the average values of ultimate strengths contained in the following table:

TABLE I*

	k_2 Tons	k_1 Square Inches	k_2/k_1
Granite (av.)	7.5	0.3	25
Glass (various sorts)	4.3–8.7	2.5–6.0	15+
Marble (av.)	4.0	0.35	11
†Bluestone	6.8	0.7	9.7
Limestone (av.)	3.5	0.5	7
Glass (common green)	10	1.5	6.7
Iron, cast	40	7.5	5.3
Zinc, cast	(10)	2.5	(4+)
Georgia Yellow Pine (across grain)	0.7	0.3	2.3
Copper, cast	(20)	12	1.7
Steel, cast	35	35	1
Copper, bolts	15	15	1
Slate	5	5	1
White Oak (across grain)	1	1	1
White Oak (with grain)	3.5	5	0.7
Iron, wrought (av. good bars)	16–20	25	0.6–0.8
Georgia Yellow Pine (with grain)	2.5	6	0.4

* All figures in this table, excepting those in the third row, from H. H. Suplee, *The Mechanical Engineer's Reference Book* (4th ed., 1913).

They represent at best only average values and are used here only in a qualitative sense. Values of K_2 given in parenthesis indicate loads producing 10 per cent reduction in original lengths.

All figures are given in the original in lbs. per sq. in.

The values for various sorts of glass are taken from Winkelmann and Schott, quoted in O. D. Chwolson, *Lehrbuch der Physik*, Vol. I (Braunschweig, 1902), pp. 709 and 712.

† "Fine-grained, compact, very strong and durable" graywacke (of Hamilton age). G. P. Merrill, *Stones for Building and Decoration* (2d ed., 1897), p. 322.

According to the formula, the angle θ will differ the more from 90° the greater the ratio of the values of crushing and tensile strength.

Correspondingly, we find that substances like glass shear at very acute angles,[1] while for mild steel the angle varies between 80° and 100°.[2]

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[1] 20°–30° in the case of glass of thick microscopic slides, according to tests made by the writer.

[2] See results of experiments by W. Mason and G. H. Gulliver, as given in W. Mason, "The Lüders' Lines on Mild Steel," *Proc. Phys. Soc. of London*, Vol. XXIII (1911), p. 306.

In the case of those substances in which the tensile strength is greater than the crushing strength, the formula would indicate that the shearing angle is obtuse. That this is indeed the case, for instance in the striking case of wood cut with the grain, may be seen from Figure 3, page 17, of Leith's *Structural Geology.*

This leads us to the important generalization that the angle of shearing of a material is the more acute the more brittle the substance is, and vice versa. In fact, it seems possible, if not probable, that in the hands of the physicist the angle of shearing will be made the chief criterion of brittleness.

The formula also brings out clearly the fact that the angle of shearing is independent of the hardness of the material. In the table given above wrought iron ranks with oak and pine wood. Small cubes of a "brittle" rubber, sold under the trade-name "soap rubber," produce shearing planes in the form of pyramids exactly like those seen in cubes of sandstone in ordinary crushing tests, with apical angles of 50° or even less. This shows that the shearing angle is also independent of the absolute amount of deformation of which the substance is capable below the elastic limit.

Strongly ductile[1] bodies, on the other hand, like soap, shear at very obtuse angles.

[1] The writer knows that in this paper he is using the word "ductile" in a sense which is sure to be severely criticized. He would be delighted to see such criticism lead to a fruitful open discussion of the fundamental conceptions involved in the deformation of solids. At this place the restricted sense in which the word "ductility" is used here may be defined best by giving it as one of several purely empirical characteristics of solids under deformation.

Substances differ in

1. The force required to produce the same absolute amount of deformation. (Small: rubber, wet clay; large: steel.)
2. The absolute amount of deformation required to reach the elastic limit. (Small: steel, wet clay; large: rubber.)
3. The percentage of any given deformation which remains permanent when the stress is removed. (0 per cent = perfect elasticity; 100 per cent = perfect plasticity.)
4. The additional force required to produce an additional amount of permanent deformation (negative, zero, positive).
5. The time required to produce the same absolute amount of permanent deformation without rupture.
6. The position of the point of rupture with reference to the yield point. (Point of rupture <or> yield point.)

In this paper a substance is called "brittle" when its point of rupture lies near its yield point. It is called "the more ductile" the farther beyond the yield point its point of rupture lies. In a "perfectly ductile" substance it lies an infinite distance beyond. In a "perfectly brittle" substance it is reached before the yield point.

When rocks are subjected to deformation in nature, the question whether they will break or bend depends largely on the degree of brittleness they possess under the given conditions and not on their hardness. This explains why geologists have been justified in the attempt to reproduce in the laboratory the various structures exhibited by the hard materials of the earth's crust by the use of soft clay, mixtures of clay and plaster of paris, and even wet sand. The use of the angle of shearing as an index of brittleness opens up new possibilities for standardizing the materials used in such experiments to accurately reproduce definite actual conditions.[1]

Mohr's formula could be quantitatively correct only, if the curve $\tau_m = f(\sigma)$ were practically a straight line between the circles of ultimate tensile and compressive stress. This, however, is not true. The form of the curve, therefore, must first be determined experimentally for each substance. From it the variable θ can be computed, from point to point by analogous formulas.

This was carried out for the first time, as far as the writer knows, in a series of excellent experiments by Kármán.[2]

He used an apparatus in which small cylinders of rock could be subjected simultaneously to hoop and longitudinal pressures in such a way that either pressure could be controlled without changing the other. The results of his experiments are embodied in stress-strain diagrams which in the most striking way show the fact that the materials used—marble and sandstone—change step by step with increasing hoop pressure from a state of perfect brittleness to one of perfect ductility. In this respect Kármán's experiments supplement beautifully the brilliant investigations of Adams.

With low hoop pressures shearing occurred in the rock cylinders resulting in the formation of Lüders' lines on the polished surfaces and, with lowest hoop pressures, leading to rupture.

In the following table[3] the observed values of the angles of shearing at various hoop pressures are placed side by side with the values computed according to Mohr's graphic construction.

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[1] The angle of shearing can readily be determined for many substances by means of an ordinary vise, if small cubes (1 cm^3) are used.

[2] Th. von Kármán, "Festigkeitsversuche unter allseitigem Druck," *Zeitschr. des Vereins deutscher Ingenieure*, Vol. LV (1911), pp. 1749–57.

[3] Tables 1 to 4 of Kármán's paper.

The agreement is sufficiently close to strongly support Mohr's theory. In addition, however, the figures reveal the striking fact that as the material changes, under the action of all-sided pressure, from a brittle to a ductile substance, the angle of shearing grows progressively less and less acute. Kărmăn's experiments have, therefore, completely verified in the case of one and the same substance the inference that the less brittle a substance is the larger is its angle of shearing.

TABLE II

	Hoop Pressure $\sigma_2=\sigma_3$ in Atmospheres	Effective Longitudinal Pressure $\sigma_1-\sigma_2$ in Atmospheres	θ Observed without Reduction*	θ Observed with Reduction	θ Computed
Marble	0 235 500 685	1360 2100 2650 2880	54° 59° 72° 83°	54° 58° 65° 70°	53° 58° 63° 73°
Sandstone	0 280 555	690 2040 2580	38° 70° 82°	38° 69° 73°	40° 63° 70°

* To find the true angle at which rupture actually took place, it is necessary to reduce the observed angle to the value it had when the rock cylinder was deformed under its load.

We can, however, go even one step farther. If growing circumferential pressure at right angles to the direction of maximum (compressive) stress increases the ductility of a substance, and with it the angle of shearing, circumferential tension must decrease it, that is, render the substance more brittle. This is completely born out by the experiments published in 1911 by W. Mason.[1] He subjected tubes of mild steel to longitudinal compressive stress simultaneously with the application of interior hydrostatic pressure, and in one series of experiments, made the tubes undergo longitudinal tension while applying water pressure externally. The angle facing the direction of maximum compressive stress, in the absence of circumferential pressure, measured[2] approximately 100°. With growing tension normal to the direction of compression, the angle changed from 100° to values as low as 84° and even[1] 79°. These values remained the same, whether the compressive stress acted longitudinally or as hoop stress.

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[1] W. Mason, "The Lüders' Lines on Mild Steel," *Proc. Phys. Soc. of London*, Vol. XXIII (1911), pp. 305–33.

[2] *Ibid.*, Table B, values at bottom of column.

Here, then, we have a substance which normally shears under compression at an angle of 100° changed to one shearing at 80° through the action of tension in all directions normal to that of the axial compression, or, using the word in the sense defined above, we may say, the substance has been made more brittle.

III. THE ELLIPSOID OF STRAIN

The attempt to correlate joint planes with stress-strain relations, to which the first part of this paper was devoted exclusively, is by no means new. Steidtman's splendid paper on "The Secondary Structures of the Eastern Part of the Baraboo Quartzite Range, Wisconsin"[2] is well known, and Leith, in his lectures and in his *Structural Geology*[3] has impressed on the younger generation of geologists the importance of shearing planes in the mechanics of rock fractures by the use of a wire-netting model.

Unfortunately, however, the model as well as the earlier discussions by other writers, give expression only to that case in which the planes of shearing form an obtuse angle in the direction of compressive stress.

The outstanding characteristic of the strain ellipsoid illustrated by Leith's wire-netting model, is the fact that the elongation in the direction of one principal stress equals the shortening in the direction of the other principal stress, or, that the area of the strained surface remains unchanged. A simple mathematical consideration shows that when a circle is changed into an ellipse of equal area, the angle of the lines of no distortion facing the direction of shortening, must always exceed 90°[3] (Fig. 14). To reduce this angle to the smaller value characteristic of all brittle materials, we must assume the longitudinal shortening to be smaller than the transverse elongation (Fig. 15), that is, increase the area of the Figure under deformation.[1]

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[1] W. Mason, "The Lüders' Lines on Mild Steel," *Proc. Phys. Soc. of London*, Vol. XXIII (1911), Table D, bottom of column (complementary angle).

[2] *Jour. Geol.*, Vol. XVIII (1910), pp. 259–70.

[3] New York: Henry Holt & Co., 1913, pp. 18–20.

This, however, leads to the conclusion that under simple non-rotational stress a brittle body suffers an increase of volume. Since the angle of shearing is the more acute the more brittle a substance is, we must expect the increase of volume under stress to be the greater the more brittle a material is.

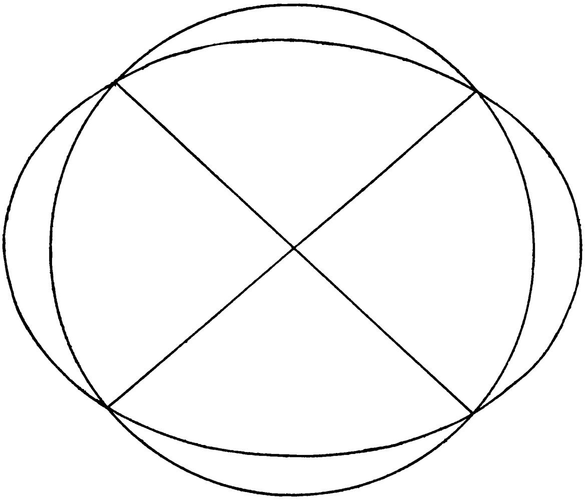

FIG. 14.—Diagram showing the position of the lines of no distortion in an ellipse of strain derived from a circle of equal volume. The angle of shearing is obtuse.

This seems indeed to be the case. Chwolson[2] gives the following formula connecting the modulus of volume increase (under tension), η, with the modulus of longitudinal strain (Young's modulus), α, and Poisson's ratio $\left(\frac{\text{lateral strain}}{\text{longitudinal strain}}\right)$, σ:

$$\eta = \alpha(1-2\sigma).$$

According to this formula, $\sigma = 0.5$, when $\eta = 0$, that is, when the volume remains unchanged during deformation. The change of volume, therefore, must be the greater the more σ differs from 0.5, that is, the smaller it is. The following values[1] of σ for different substances completely confirm the inference drawn from the graphic representation of the strain ellipsoid.

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[1] For the mathematical proof of this statement the writer is again under obligation to Dr. Brand.

[2] O. D. Chwolson, *Lehrbuch der Physik*, Vol. I, p. 713.

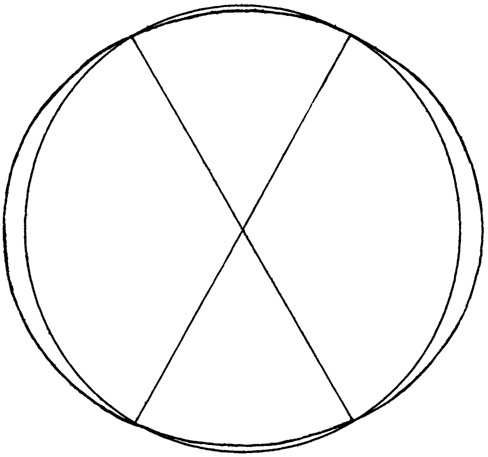

FIG. 15.—Diagram showing the position of the lines of no distortion in an ellipse derived from a circle of smaller volume (increase of volume accompanies deformation). The angle of shearing is acute.

Paraffin	0.50
Caoutchouc	0.50
Copper	0.348
Mild Steel	0.304
Iron	0.243 to 0.310
Zinc	0.205
Glass	0.197 to 0.319

That brittle bodies suffer an increase of volume when deformed under tension, is well known. That such an increase of volume also actually accompanies deformation under one-sided compression, as demanded by the graphic construction of the strain ellipsoid, seems to be proved by the experiments made by Kahlbaum and Seidler[2] and more recently by Lea and Thomas.[3]

It is essential, therefore, before we use the strain ellipsoid for the interpretation of shearing planes in nature, that we decide which form of the ellipsoid corresponds to the conditions of the specific case.

IV. PLANES OF SHEARING PRODUCED BY IRROTATIONAL AND ROTATIONAL STRAINS

We may now return to the interpretation of planes of shearing observed in nature. We have learned that Hartmann's law applies to brittle substances only, that is, that only in brittle materials the acute angle of shearing planes faces the direction of the compressive stress. We may now extend the law by adding, that in ductile substances it is the obtuse angle that faces the direction of the compressive stress.

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[1] O. D. Chwolson, *Lehrbuch der Physik*, Vol. I, p. 714.

[2] R. Kahlbaum and Seidler, *Zeitschr. Anorg. Chem.* (1902), pp. 29–30, 254–94.

[3] F. C. Lea and W. N. Thomas, "Change in Density of Mild Steel Strained by Compression beyond the Yield-Point," *Engineering*, Vol. C (1915), pp. 1–3.

Before attempting to apply the law to any specific case, therefore, we must decide whether the material under the given conditions had the properties of a brittle or those of a ductile substance. On the other hand, when the direction of the greatest principal stress is known, the position of the joint planes produced by it may be used to determine the degree of ductility which the material possessed at the time of shearing.

All the cases so far discussed involve irrotational strains only. The arrangement of shearing planes due to rotational strains, as illustrated by Leith's wire-netting model and discussed in his book on *Structural Geology*, is, of course, only possible in ductile substances, as a glance at the angle of the shearing planes will show. This model has, however, been applied successfully to some striking cases of jointing in quartzites.

We may approach the problem involved in these interesting cases by turning to an illustration in Van Hise's "Principles of North American Pre-Cambrian Geology," page 652.[1] Figure 131 shows layers of quartzite alternating with thin beds of more slaty character. The harder beds are traversed by two systems of intersecting joints, both forming angles of 50°–70° with the bedding planes, that is, forming acute angles of approximately 60° facing the bedding planes.

In the intercalated slaty beds, however, only one of these two joint systems is developed. It consists of more numerous joints inclined but 20° or less to the bedding planes. If the complementary symmetrical set were developed, the angle formed by the two systems facing the bedding planes would be 130° in these less brittle slaty beds, instead of 60° as in the brittle purer beds.

From this relation of the shearing angles in the two types of rock it is evident that the joints in the more brittle beds are due to the normal component of the stress acting on the beds. They must, therefore, have been developed essentially through irrotational strain.

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[1] *Sixteenth Ann. Rept. U.S. Geol. Survey,* Part I (1896), p. 652. See also C. K. Leith, "Rock Cleavage," *U.S. Geol. Survey, Bull. 239* (1905), p. 123, Fig. 37.

The differential movement between adjoining beds required by the structural relations indicated in the text, was obviously largely limited to slippage parallel to the bedding within the thin slaty layers. It is important to note, however, that the direction of movement here, as in several cases referred to in the first part of this paper, has influenced the number and nature of the two opposed diagonal joint systems in the brittle beds. Those inclined in the direction of drag produced by the differential movement are more numerous, closed, and slickensided, while the opposite set is represented by few and gaping joints.

The closely spaced joints in the slaty beds, on the other hand, may partly be due to true rotational strain.

Essentially the same considerations apply to the case discussed in detail by Steidtmann[1] and by Leith in his *Structural Geology*. Here the joint system opposed to the "drag" action of the differential movement between the beds is represented by but a few "open gashes or tension joints." But their presence is sufficient to indicate that in the center of the quartzite beds the effect of rotational strain has been entirely subordinate to that of the irrotational strain produced by the component normal to the bedding planes.

The writer has the suspicion that this will be found to be true most generally in bedded rocks, in which differential movement between the beds is largely accomplished by slippage along bedding planes chiefly within layers of soft rock acting as lubricants.

V. PLANES OF SHEARING IN SHALES

The absolute and relative values of the crushing and tensile strengths of a rock play a fundamental roll in the deformation of rocks along the face of natural and artificial excavations.[2]

When the depth of an excavation has reached the point at which the vertical component of the stress set up by the removal of

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[1] *Jour. Geol.*, Vol. XVIII (1910), p. 261, Fig. 1.

[2] See the excellent analysis of the factors involved in D. F. MacDonald, "Excavation Deformations," *Congrès géologique international, Compte rendu de la XII^e session* (Canada, 1913), pp. 779–92.

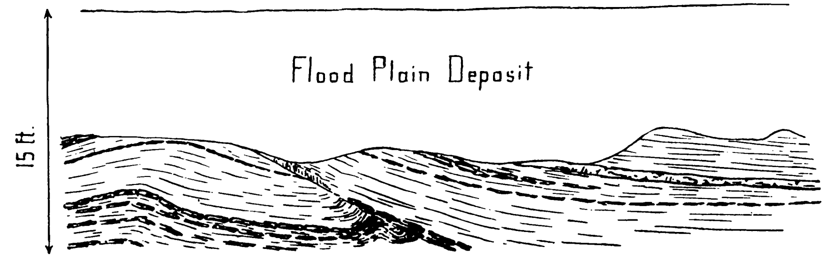

FIG. 16.—Fold and thrust fault in Eden shales, West Fork Creek, Cincinnati, Ohio

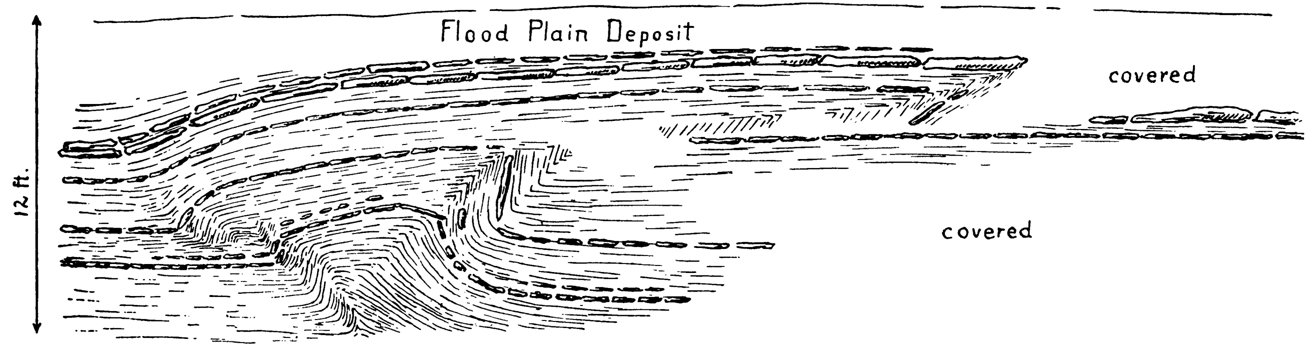

FIG. 17.—Fold and thrust fault in Eden shales, West Fork Creek, Cincinnati, Ohio

material at the toe of the new steep slope exceeds the elastic limit, rupture along planes of shearing will take place sooner or later in brittle materials. Ductile substances, on the other hand, such as clays, will flow, causing the lower part of the steep slope to bulge out and the bottom of the excavation to buckle up.[1]

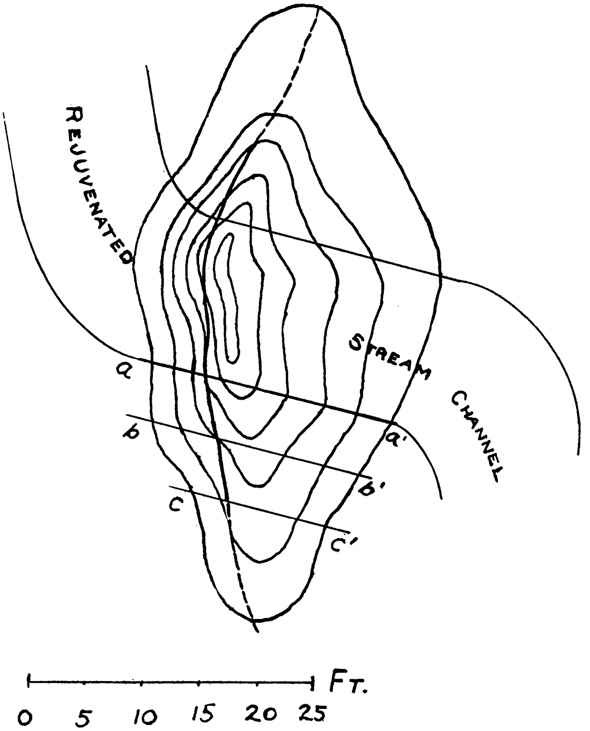

FIG. 18.—Diagrammatic contour map of the surface of a bed of shale buckled and cut by an inclined joint plane. Superimposed on it is the outline of a new stream bed cut into this structure after rejuvenation.

Van Horn has given a detailed description of this buckling at the base of a rock slide.[2] Small and entirely local anticlines which quite obviously owe their origin to this process, are of common occurrence in the bottoms of ravines cut into clays or shales, and are often directly associated with landslip terraces.[1] Identical surficial anticlinal buckles[2] which have been observed under a cover of glacial drift without any connection with steep slopes or landslides, probably owe their origin to similar stress relations resulting from the cracking or other deformation of the Pleistocene ice cover.

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[1] D. F. MacDonald, "Excavation Deformations," *Congrès géologique international, Compte rendu de la XII^e session* (Canada, 1913) p. 791, Fig. 3.

[2] Frank R. Van Horn, "Landslide Accompanied by Buckling, and Its Relation to Local Anticlinal Folds," *Bull. Geol. Soc. Amer.*, Vol. XX (1910), pp. 625–32.

In the vicinity of Cincinnati, wherever the most recent rejuvenation has cut into the bottom of ravines within the Eden shales, similar bucklings are quite common.[3] Frequently, however, these anticlines are not only overturned, but faulted, generally in the form of a miniature overthrust, such as shown in the Figures 16 and 17.

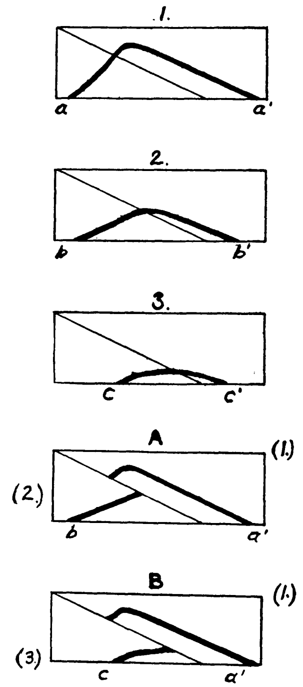

Fig. 19.—Sec. 1 shows the structure exposed on the south side of the stream channel shown in Fig. 18. Secs. 2 and 3 are drawn parallel to and 5 and 10 feet, respectively, south of, sec. 1. Sec. A shows the result of squeezing out the shale layers from underneath the joint plane in such a way that the lower portion of sec. 2 is pushed out so as to rest under the upper portion of sec. 1. In sec. B this process is carried farther, the lower portion of 3 pushed out so as to rest underneath the upper portion of 1.

In the light of the preceding discussion, it appears highly probable that these miniature "reversed faults" have nothing to do with horizontal compressive stresses. The shales are distinctly ductile, as is implied by the very existence of the anticlines due to flowage. They are, however, ductile only to a limited degree. After the strain has reached a certain limit, they rupture along planes of shearing. Here the obtuse angle formed by two symmetrical joint planes points to the action of compressive stress in a perpendicular direction, or especially to the horizontal tensile stress which results when the upper layers are forced up by the pressure of the flowing layers of shale underneath.

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[1] E.g., T. C. Hopkins and W. M. Smallwood, "Discussion of the Origin of Some Anticlinal Folds near Meadville, Pennsylvania," *Bull. Syracuse University*, Ser. IV, No. 1, 18 (quoted from Van Horn, *loc. cit.*).

[2] For instance, G. K. Gilbert, "Dislocation at Thirtymile Point, New York," *Bull. Geol. Soc. Amer.*, Vol. X (1899), pp. 131–34; F. R. Van Horn, "Local Anticlines in the Chagrin Shales at Cleveland, Ohio," *ibid.*, Vol. XXI (1910), pp. 771–73.

[3] E. L. Braun, "The Cincinnatian Series and Its Brachiopods in the Vicinity of Cincinnati," *Jour. Cinc. Soc. Nat. Hist.*, Vol. XXII (1916), No. 1.

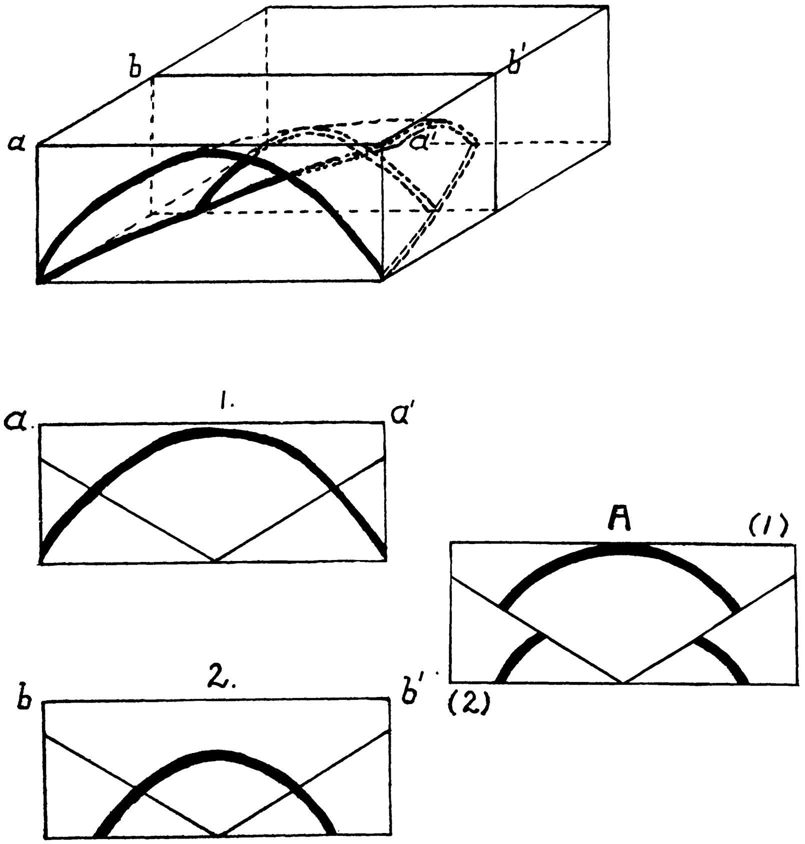

FIG. 20.—Block diagram representing the pitching end of an anticline.
Sec. a–a′ = sec. 1 below.
Sec. b–b′ = sec. 2 below.

These sections show, in addition, the position of two inclined joint planes striking parallel to the axis of the fold. Sec. A shows the result of squeezing out the shale layers from underneath the joint plane in such a way that the lower portion of sec. 2 is pushed out so as to rest under the upper portion of sec. 1.

Whenever a stream removes a portion of such an anticline, as indicated on the map sketch in Figure 18, the tendency exists to squeeze the dipping beds up from underneath the joint plane, out into the channel.

Sections A and B (Fig. 19) show the result which is brought about entirely by a horizontal flowage of the beds beneath the joint plane in a direction perpendicular to the plane of the paper, and not to any horizontal compressive stresses acting within the plane of the paper.

The joint nature of these "fault" planes is brought out beautifully by the exposure shown in Figure 17, which for years has been well exposed in Westfork Creek at Cincinnati, just above the schoolhouse.

There can be little doubt that essentially the same interpretation applies to other similar occurrences.[1]

VI. HORIZONTAL COMPRESSIVE STRAINS IN GRANITE

Compressive strains of considerable magnitude, essentially in a horizontal direction, exist at widely separated localities in all states of New England, if not throughout the whole region. In the quarries, the strains find expression in various ways. Vertical drill holes are flattened to an elliptical cross-section,[2] the cores between contiguous borings are crushed, and cracks open up diagonally from the channels.[3] In the process of quarrying new fissures open up with a dull explosive noise,[4] or new sheetlike partings form[5] or old sheets buckle up.[6]

With a detailed knowledge of most of the excellent exposures of sheeted granite in New England at his command, Dale has come to the conclusion that shrinkage in cooling, or changes of temperature, or other forms of weathering have played, at best, only a secondary rôle in the production of the sheet structure; that the main factor has been horizontal compressive stress such as now finds expression in the region.[1]

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[1] A. H. Purdue, "Illustrated Note on a Miniature Overthrust Fault and Anticline," *Jour. Geol.*, Vol. IX (1901), pp. 341–42; C. E. Decker, unpublished manuscript, see Fig. 17, p. 39, *Jour. Geol.*, Vol. XXVI (1918).

[2] T. N. Dale, "The Granites of Vermont," *U.S. Geol. Survey, Bull. 404*, p. 18, Fig. 2; *Bull. 354*, p. 34.

[3] *Ibid., Bull. 354*, pp. 96 and 126; *Bull. 313*, pp. 12 and 142.

[4] *Ibid., Bull. 313*, pp. 34 and 142.

[5] *Ibid., Bull. 404*, pp. 97 and 107.

[6] *Ibid., Bull. 313*, Pl. VII, A.

This is not the place nor the time to enter into a discussion of this complex problem. The final word has not yet been spoken.

If we assume, for the time being, a diastrophic origin of the observed strains, we can point to three observations favoring such a view.

1. For at least one locality, the famous Quincy quarries south of Boston, Dale records the observation that "this strain in some quarries appears to increase with their depth."[2]

2. At Fletcher Quarry, on Robeson Mountain, in Washington County, Vermont, Dale observed what he called "double-sheet" structure. Here, instead of the usual single set of sheets, two such sets, intersecting at an angle of about 42° are exposed. If we analyze their position according to the method described in the first part of this paper we find that the bisectrix of the acute angle, that is, the direction of the greatest principal (compressive) stress, trends N. 60 W.–S. 60 E. and that it differs but slightly from the horizontal, being slightly directed downward toward the southeast. The least (tensile) principal stress, on the other hand, is practically directed upward, in the direction of easiest relief from the horizontal pressure.[3]

Fortunately, there is, at the same locality, "a marked northeast–southwest compressive strain in the upper part of the quarry, raising the sheets and even forming new sheet partings." The direction of this strain is essentially that which would result from the compressive stress, acting from the northwest, inferred from the "double-sheets." While this may, of course, be a mere coincidence, it certainly is suggestive of a causal connection.

None of the other quite numerous cases in which Dale records the direction of compressive strain, together with data concerning the position of the sheet structure, can be used to test this matter further. For with only one of the two intersecting planes of shearing given, we cannot even guess at the position of the principal stresses.[1]

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[1] T. N. Dale, "The Granites of Connecticut," *U.S. Geol. Survey, Bull. 484* (1911), pp. 29–36.

[2] *U.S. Geol. Survey, Bull. 354*, p. 96.

[3] If we had reason to believe that the granite had been in a ductile state when these planes of parting were formed, the stresses would have to be reversed. But all observations seem to speak against this possibility.

3. The New England region is that part of the United States of which we know most definitely and quantitatively that it has undergone considerable deformation in post-glacial time. A glance at Fairchild's map of "Isobases of Post-glacial Uplift"[2] shows that if we can assume the earth's surface to the south and southeast to be relatively at rest and the domelike upheaval with its center halfway between Quebec and James Bay to be rising independently, there should exist a compressive stress in the general direction northwest–southeast throughout the surface of New England. It may, of course, be mere coincidence that this is the same direction which we found required to produce the "double-sheet" structure on Robeson Mountain. But, combined, these three observations are distinctly favorable to the view that the larger part of the remarkable sheet structure of the granites of New England is due to compressive stresses arising from the larger earth movements indicated by Fairchild's isobasic maps.

The objection may be raised that in quarries in other parts of New England strains in different directions have been observed.

Great variations in local conditions of stress are not surprising in view of two important facts. The direction of relief in these cases is normal to the surface. Where the surface is far from horizontal the position of one of the two principal stresses must vary from place to place, and with it, the position of the resulting plains of shearing.

Furthermore, observation shows that the granites, which exhibit the sheet structure, are divided by various joint systems, some unquestionably of earlier origin. It is, therefore, not an unbroken sheet of granite that is being deformed in this region, but a mosaic of large and small blocks. The stress conditions, resulting from such a complex arrangement, must be expected to vary considerably from place to place, while corresponding only in the aggregate to the simple fundamental stress relation.[1]

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[1] If, for instance, in the case illustrated in Fletcher Quarry only the planes striking N. 20 W. had been given, we would probably not have considered them to be in harmony with a compressive stress acting from the northwest.

[2] H. L. Fairchild, "Post-glacial Uplift of Northeastern America," *Bull. Geol. Soc. Amer.*, Vol. XXIX (1918), p. 202.

Whatever may be the true merits of this interpretation of the "double-sheet" structure, it serves well to bring out the fact that horizontal compressive stresses will produce planes of shearing at low angles only in highly brittle substances close to the surface, where the least principal stress is directed upward, in the direction of easiest relief.

VII. LOW-ANGLE FAULTING

This leads us finally to the important question of low-angle faulting which has recently been given a most valuable discussion by R. T. Chamberlin and W. Z. Miller.[2]

The heart of the problem is nowhere clearer exposed than in the northwest Highlands of Scotland.[3] A series of sediments of diverse nature, piled up, shingle-like, by thrust faults inclined on the average about 45°, has been cut across by several practically horizontal planes of fracture, along which the individual slices have been moved in the general direction of thrust for many miles.

The strongly inclined faults clearly are planes of pure (irrotational) shear produced by the horizontal compressive stress. The question is, Can the essentially horizontal planes of fracture have been produced under similar conditions of strain or do they represent the result of rotational strain?

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[1] For a similar reason, the movement on individual shearing planes, into which a block has been broken under one-sided pressure, may be most diverse, one block being pushed out in one direction, the other in another. This explains the small success that has attended the attempts made by Dr. Salomon (1911) and some of his students (Lind, 1910, Dinu, 1912, Spitz, 1913, Engstler, 1913, Seitz, 1917) to determine the nature of the stress involved in the formation of joint systems from the direction of movement recorded on the slickensides. (For complete references to these most careful studies, see O. Seitz., *Über die Tectonik der Luganer Alpen*, Inaug. Diss., Heidelberg, 1917).

The same criticism applies equally to the attempts made to infer from local observations of movements during an earthquake the nature of the major stresses which have resulted in the formation of great fault lines. The cause of such movements along existing fault lines may be fundamentally different from that which caused the fracture itself.

[2] R. T. Chamberlin and W. Z. Miller, "Low-Angle Faulting," *Jour. Geol.*, Vol. XXVI (1918), pp. 1–44.

[3] B. N. Peach, John Horne, W. Gunn, C. T. Clough, and L. W. Hinxman, "The Geological Structure of the North-West Highlands of Scotland," *Mem. Geol. Survey of Great Britain*, 1907.

To prove the first assumption, we must find a cause for the great change in inclination of the shearing planes from rather high angles to a horizontal position. Cadell's experiments and their own researches led Chamberlin and Miller to the suggestion that the additional vertical pressure resulting from the piling up of numerous thrust-blocks may tend to reduce the angle of shearing.

In the light of the preceding discussion it appears that an additional load would rather have the opposite effect. It would be equivalent to increasing the hoop pressure in Kármán's experiments. It would render the rock less brittle and therewith the angle of shearing larger. Since the compressive stress, in this case, acts in a horizontal direction this would mean planes of shearing inclined at a steeper angle than before.

Any given rock shows the smallest angle of shearing at the surface. To reduce this angle still more, an active tensile stress would have to be applied in a vertical direction, which is, of course, out of question.

In a general way, then, we must picture to ourselves the planes of pure shear produced in the earth's crust by a horizontal compressive stress as being least inclined near the surface and becoming progressively steeper at lower levels so as to approach finally the more or less vertical position of the planes of flowage existing at considerable depths. Conceptions such as are expressed, for instance, in Ulrich's diagrams to illustrate the "inland migration of belts of folding in Southeastern North America" in his "Revision of the Paleozoic Systems"[1] require corresponding modification.

For an understanding of the mechanics of the large horizontal overthrusts, however, we can but turn to the only alternative, which was clearly brought out by Chamberlin and Miller, that, in contrast to the other type, they are essentially the result of rotational strains of the kind which Becker has termed "scission," which occur "when a bar or plate is shorn by a pair of shears, or when a rivet yields perpendicularly to its axis, say, in a bursting boiler."[1]

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[1] *Bull. Geol. Soc. Amer.*, Vol. XXII (1911), p. 441.

A clear and simple illustration of the form in which this method of fracturing finds its most important expression in the earth's structure, is offered by Figure 21, which represents the section of one of the folds of the Jura Mountains at the boundary of France and Switzerland, as brought out by Buxtorf's detailed investigations.[2] The brittle layer of the Rauracien, which stood up vertically in the process of folding, comparable to the rivet on a boiler plate mentioned above, gave way to the pressure of the swelling plastic Oxford clays, counteracted below by the level part of the north limb of the anticline.

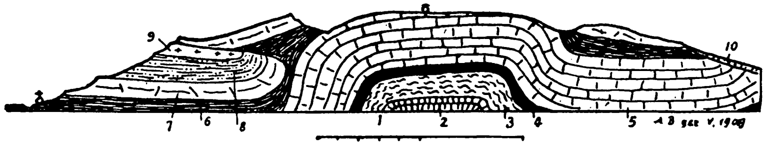

FIG. 21—Cross-section of a fold of the Jura Mountains at the border of France and Switzerland (at the "Clos du Doubs"). 1-3, Triassic, 4. Liassic, 5. Dogger, 6. Oxford clays, 7. Rauracian, Sequanian, 9. Kimmeridgian, 10. Landslide. (A. Buxtorf, 1909.)

The writer is inclined to believe, following the lead of Dr. Buxtorf,[3] that in the case of most, if not all "low-angle faults," or as Suess[4] called them, "listric planes," *the rotational stress resulted from the flowage of materials inside the fold*, whether it be, on a smaller scale, confined to individual normally plastic layers, or, on the largest scale, to the forced flowage of the rocks forming the cores of rising mountains.

This, however, leads us beyond the scope of this paper. The necessary setting for a broader discussion of this important problem will be given in a paper now in preparation.[5]

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[1] G. F. Becker, "Finite Homogeneous Strain, Flow and Rupture of Rocks," *Bull. Geol. Soc.*, Vol. IV (1893), p. 24.

[2] A. Buxtorf, "Über den Gebirgsbau des Clos du Doubs und der Vellerat-Kette im Berner Jura," *Ber. Vers. Oberrhein. Geol. Ver.* (1909), p. 76, Fig. 1.

[3] *Ibid.*, p. 86.

[4] E. Suess, *The Face of the Earth* (transl. by Sollas, H. B.), Vol. IV (1909), p. 536.

[5] Parts of which were presented before the Geological Society of America at the Chicago meeting 1920 under the title "The Probable Cause of the Localization of the Major Geosynclines."